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Ising model at an edge: a position space renormalisation group approach

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Abstract. The critical behaviour of a three-dimensional Ising model, bounded by two plane surfaces meeting at an angle α, is studied using the Migdal-Kadanoff and two-terminal cluster renormalisation methods. We obtain expressions for the edge fixed point and for the independent edge exponent y_{he} at the various transitions. We find that y_{he} is not completely universal but depends on the angle α, and that it shows some unexpected features.

1. Introduction

The Ising model with a free surface has been extensively studied, both with field-theoretic methods and with several renormalisation group (RG) methods in position space. A recent review on this subject is given by Binder (1983). Also the Migdal-Kadanoff RG (MKRG, Migdal 1975, Kadanoff 1976) has been successfully applied to this problem (Lipowsky and Wagner 1981), providing a qualitatively correct picture although the numerical estimates of fixed points and critical exponents are not very reliable. In more than two dimensions four different types of phase transitions can occur on the surface: the ordinary, the surface, the extraordinary and the multicritical special transitions. In two dimensions only the bulk-driven ordinary transition can take place, since the surface is one dimensional in this case.

Recently, Cardy (1983) considered the critical behaviour in the vicinity of a free edge, where two planar surfaces intersect. Edge critical exponents, which describe how quantities close to the edge diverge when the system approaches one of its critical points, are related through scaling laws to bulk and surface critical exponents, as well as to one new edge magnetic scaling power, y_{he}. This exponent depends on the edge interaction and magnetic fields, in addition to bulk and surface quantities.

The purpose of this paper is to present results on edge critical behaviour obtained using position space RG methods. It is organised in the following fashion: in § 2 we write down the model and in § 3 the RG calculations are outlined. We apply both the MKRG and a two-terminal cluster (TTC) RG developed by Tsallis and Levy (1981). The fourth section contains our results. It turns out that a thermal-like perturbation close to the edge is irrelevant, which is natural since the edge is effectively one dimensional. However, as mentioned previously, an edge magnetic field is relevant in almost all cases. When the surface undergoes an ordinary or special transition, the edge exponents are not completely universal, but they depend on the angle α between the two planes which define the edge. This is not the case at the surface and extraordinary transitions, however. Our calculations revealed two unexpected features. Firstly, for $\alpha < \alpha^* \approx 53°$

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at the ordinary transition, the edge fixed point disappears while the edge exponent becomes negative, indicating that a magnetic perturbation close to the edge no longer changes the critical behaviour. Secondly, we find that the edge fixed point diverges when $\alpha > \alpha^{**} \approx 297°$, and the surface-bulk system undergoes a special transition. This implies that the edge transition is first order, at least within the MKRG and TTC RG approximations.

Finally, the last section contains our conclusions.

2. Model

Consider an Ising model on a three-dimensional lattice. In one direction, denoted by z, its extension is infinite, while the spins in the remaining two directions are restricted to lie in the region $0 < \phi < \alpha$, where ϕ is the polar angle and $0 < \alpha < 2\pi$. For $\alpha = \pi/2$, π or $3\pi/2$, we take the lattice to be simple cubic, while for angles that are multiples of $\pi/3$ we assume a hexagonal lattice. In particular, when $\alpha = \pi$, we have two different types of lattice, which might yield different results.

The Hamiltonian, in units of $k_B T$, is given by

$$H[s_i] = \sum_{\langle ij \rangle} B(s_i, s_j) \tag{2.1}$$

where

$$B(s_i, s_j) = \begin{cases} K_b s_i s_j + h_b(s_i + s_j)/z_{bb} & s_i, s_j \in \Lambda - \Lambda_s \\ K_b s_i s_j + h_b s_j / z_{bb} & s_i \in \Lambda_s \quad s_j \in \Lambda - \Lambda_s \\ K_s s_i s_j + h_s(s_i + s_j)/z_{ss} & s_i, s_j \in \Lambda_s - \Lambda_e \\ K_s s_i s_j + h_s s_j / z_{ss} & s_i \in \Lambda_e \quad s_j \in \Lambda_s - \Lambda_e \\ K_e s_i s_j + h_e(s_i + s_j)/z_{ee} & s_i, s_j \in \Lambda_e \end{cases} \tag{2.2}$$

$\langle ij \rangle$ indicates the sum over nearest-neighbour pairs, Λ_e is the set of edge sites, Λ_s is the set of surface sites (including Λ_e) and Λ contains all sites. The K and h are nearest-neighbour coupling constants and magnetic fields, respectively, and subscripts b, s and e refer to bulk, surface and edge quantities. In the following we will often denote these indices collectively with the label p, and we will also suppress the subscript b for bulk quantities, when this can be done unambiguously. We have chosen to split up each one-spin interaction on the links that connect the site with its nearest neighbours of the same type. Here z_{bb} is the number of bulk nearest neighbours to a particular bulk spin, and z_{ss} and z_{ee} are the corresponding surface and edge numbers. In addition, we introduce z_{bs}, the number of bulk neighbours to a surface site, and z_{be} and z_{se} analogously. For the cubic lattice, $z_{bb} = 6$, $z_{ss} = 4$, $z_{ee} = 2$, $z_{bs} = 1$, $z_{se} = 2$, $z_{be} = (2\alpha/\pi - 1)$, while in the hexagonal case, $z_{bb} = 8$, $z_{ss} = 4$, $z_{ee} = 2$, $z_{bs} = 2$, $z_{se} = 2$, $z_{be} = (3\alpha/\pi - 1)$.

In principle, in addition to the interactions considered here, we could also have allowed for different coupling constants and fields some distance away from the boundary, or perpendicular to it, but it turns out that these renormalise to the same fixed point as the bulk K and h, at least when we apply the MKRG. Hence they represent irrelevant perturbations which can be ignored from the outset.

3. Renormalisation group methods

3.1. Migdal-Kadanoff RG

We now apply the MKRG to the model above. This scheme consists of two steps: first

a fraction of the bonds connecting the lattice sites are moved to produce a new Hamiltonian $\tilde{H}$, and then an exact renormalisation transformation is performed on the bond-moved model. $\tilde{H}$ is defined on a decorated lattice $\tilde{\Lambda}$, which is isomorphic to the original lattice Λ, but with a lattice spacing that is b times larger. Each bond K_p that does not belong to the decorated lattice is moved, perpendicular to itself, to the closest link in $\tilde{\Lambda}$. If there are several links in $\tilde{\Lambda}$ at the same distance from K_p, we divide the increased strength equally between these bonds. Although there is no *a priori* reason for this choice, it appears to us to be the most natural one, and it reproduces the recursion relations of Lipowsky and Wagner in the infinitesimal rescaling limit. Thus every link in $\tilde{\Lambda}$ is decorated with $b-1$ spins, connected by enhanced interactions $\tilde{K}_p$. This procedure is illustrated in figure 1, where bonds parallel to the z axis are moved

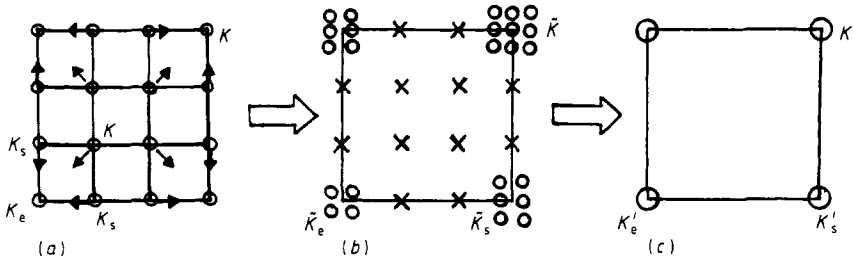

Figure 1. Illustration of the MKRG procedure on a cubic lattice, close to an edge with $\alpha = 90°$. The rescaling factor is 3. In the first step, $(a) \rightarrow (b)$, bonds parallel to the z axis (perpendicular to the plane of the page) are moved to the closest link in the decorated lattice. Surface and bulk bonds also get contributions from adjacent cells. The second step $(b) \rightarrow (c)$ consists of exact decimation on the bond-moved lattice.

in the xy plane. We find

$$\tilde{K} = b^2 K \tag{3.1a}$$

$$\tilde{K}_s = bK_s + (b-1)bK/2. \tag{3.1b}$$

Bond-moving could only be applied to lattices where α is a multiple of $\pi/3$ (hexagonal) or $\pi/2$ (cubic). In both cases, the decorated edge interaction is given by

$$\tilde{K}_e = K_e + (b-1)K_s + (b-1)[(b+1)\alpha/2\pi - \tfrac{1}{2}]K. \tag{3.1c}$$

We propose to use this formula for arbitrary α, although its validity is questionable expecially for α close to 0 or 2π. Note that for $\alpha = \pi$, and $K_e = K_s$, $\tilde{K}_s$ from (3.1b) and $\tilde{K}_e$ from (3.1c) agree, as they have to do. It is also remarkable that the bond-moving procedure yields the same result for both types of lattice. It is a well known artefact of the MKRG for the bulk that it does not distinguish between different types of lattice, and obviously this misfeature extends to surface and edge interactions.

The next step in the MKRG is a dedecoration transformation, in which a partial trace is performed over the $b-1$ spins that decorate each link of $\tilde{\Lambda}$. This is easily accomplished, yielding

$$\tanh K'_p = (\tanh \tilde{K}_p)^b \qquad p = b, s, e. \tag{3.2}$$

In the infinitesimal rescaling limit, $b = 1 + \delta\lambda$, we obtain the following differential

recursion relations:

$$dK_p/d\lambda = \bar{K}_p + \phi(K_p)/2 \tag{3.3}$$

where

$$\phi(K_p) = \sinh(2K_p)\ln(\tanh K_p) \tag{3.4}$$

and

$$\bar{K} = 2K \tag{3.5a}$$

$$\bar{K}_s = K_s + K/2 \tag{3.5b}$$

$$\bar{K}_e = K_s + (\alpha/\pi - \tfrac{1}{2})K. \tag{3.5c}$$

Note that for $K = K_s = 0$, the recursion relation for K_e reduces to the MKRG equation for the bulk in one dimension. As usual, the fixed points of (3.3)–(3.5), which we denote by K_p^*, correspond to the critical temperatures, and the thermal scaling power is defined by

$$y_{tp} = \frac{\partial}{\partial K_p}\left(\frac{dK_p}{d\lambda}\right) = \frac{\partial K_p}{\partial K_p} + \frac{\phi'(K_p)}{2}. \tag{3.6}$$

As indicated in (2.2), the magnetic field h_p at one particular site is divided equally between the z_{pp} bonds of type p that emerge from that site. The resulting interaction terms, $(h_p/z_{pp})(s_i + s_j)$, of the Hamiltonian are moved along with the nearest-neighbour interaction. For an infinitesimal rescaling this procedure yields to first order in the magnetic fields:

$$\partial h/\partial\lambda = \bar{h} - \phi(K)h \tag{3.7a}$$

$$\partial h_s/\partial\lambda = (\bar{h}_s - \phi(K_s)h_s) + (\bar{h} - \phi(K)h)z_{bs}/x_{bb} \tag{3.7b}$$

$$\partial h_e/\partial\lambda = (\bar{h}_e - \phi(K_e)h_e) + (\bar{h}_s - \phi(K_s)h_s)z_{se}/z_{ss} + (\bar{h} - \phi(K)h)z_{be}/z_{bb} \tag{3.7c}$$

where

$$\bar{h} = 2h \tag{3.8a}$$

$$\bar{h}_s = h_s + h/2 \tag{3.8b}$$

$$\bar{h}_e = h_s + (\alpha/\pi - \tfrac{1}{2})h. \tag{3.8c}$$

The magnetic eigenvalue y_{hp} is obtained in the usual fashion, by differentiating $\partial h_p/\partial\lambda$ with respect to h_p at the fixed point. In particular, because $\bar{K}_e$ and $\bar{h}_e$ do not depend on K_e in three dimensions, we can obtain a very simple expression for y_{he}

$$y_{he} = -\phi(K_e^*) = 2[K_s^* + (\alpha/\pi - \tfrac{1}{2})K^*] \tag{3.9}$$

valid provided K_e^* is finite.

3.2. Two-terminal cluster RG

In order to test some of the results obtained with the MKRG, we also applied an alternative position space RG recipe, namely the two-terminal cluster (TTC) approximation with rescaling factor $b = 2$. For the hexagonal lattice, the cluster for the renormalised interaction K' is shown in figure 2(*a*), where the spins s_1, s_2 and s_3 are traced

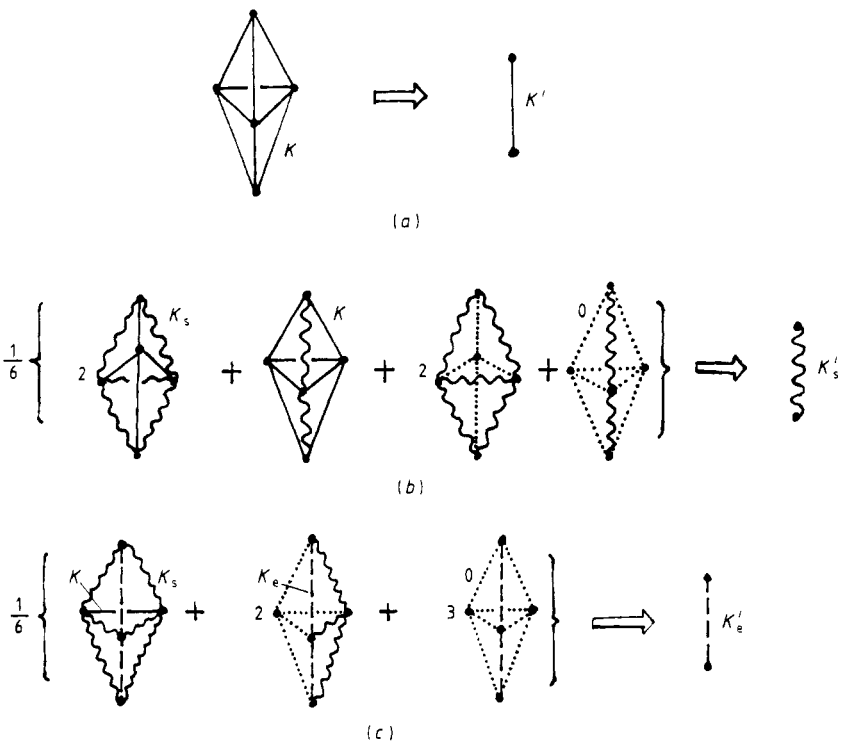

Figure 2. (*a*) TTC cell used to renormalise bulk bonds on a hexagonal lattice. The three internal spins are traced over using the 'break-collapse' method. (*b*) Combination of cells used to renormalise the surface interaction. (*c*) Combination of cells used to renormalise the edge interaction when $\alpha = 60°$. For other values of the opening angle, differently weighted averages are appropriate. —— indicates a bulk coupling, ∿∿∿ a surface coupling, - - - an edge coupling and ..... indicates the absence of an interaction. On a cubic lattice, we instead use cells with four internal spins.

over. There is a problem connected with the fact that the links in the triangular-lattice planes (the horizontal links) are not equivalent to the (vertical) links which connect these planes. We do not know of any *a priori* reasons for these two types of coupling constants to renormalise to the same value. On the other hand, it is difficult to find a proper renormalisation prescription for the horizontal interactions. The results obtained by equalising them to their vertical counterparts appear to be sensible, and therefore we do so. To renormalise the surface and edge interactions, we take weighted averages over the different possible clusters, as indicated in figures 2(*b*) and (*c*). The decimation was carried out using the break-collapse method (Tsallis and Levy 1981), which easily yields recursion relations for the K_p. The cubic lattice was analogously renormalised using clusters with four internal spins instead of three. In this case no problem about how to renormalise the horizontal interactions arises, since they are completely equivalent to the vertical ones. For bulk and surface couplings on the cubic lattice, our equations are identical to those of Lam and Zhang (1983). The recursion relation for the edge on a hexagonal lattice is given in the appendix.

There is no obvious way to handle symmetry-breaking fields with the break-collapse method. Since we are mainly interested in checking our MKRG results, we limit ourselves

to a calculation of the various fixed points and the associated thermal eigenvalues, and we do not determine the magnetic scaling powers.

4. Fixed points and exponents

We now consider the fixed points K_p^* of the MKRG equations (3.3)-(3.5). As is well known, there is one non-trivial fixed point of the bulk equation, and we found $K^* = 0.1398$ (hex: 0.24, cub: 0.18), with the associated thermal scaling power $y_t = 0.9482$ (hex: 1.11, cub: 1.17). Here and henceforth results obtained with the TTC RG are presented within parentheses, and 'hex' and 'cub' refer to hexagonal and simple cubic lattices respectively. Only one MKRG result is quoted, because both types of lattice yield the same recursion relations, as discussed above. At the above value of K, the surface relation, (3.3)+(3.4)+(3.5*b*), has two finite fixed points, $K_s^* = 0.0266$ (hex: 0.049, cub: 0.047) and $K_s^* = 0.3394$ (hex: 0.47, cub: 0.35), which correspond to the ordinary and special fixed points, respectively. We found the scaling power connected with the surface interaction to be $y_{ts} = -1.6331$ (hex: -1.91, cub: -1.85) at the ordinary fixed point and $y_{ts} = 0.6143$ (hex: 0.56, cub: 0.71) at the special point. The surface transition is governed by the fixed point at $K^* = 0$, $K_s^* = 0.4407$ (hex: 0.65, cub: 0.44), with associated eigenvalue $y_{ts} = 0.7535$ (hex: 0.52, cub: 0.87). Finally, the extraordinary transition corresponds to $K^* = 0.1398$, $K_s^* = \infty$.

The edge critical exponents are connected through scaling laws to the bulk and surface exponents, and to one new independent edge exponent, y_{he}. These scaling laws were derived by Cardy (1983), so it is not necessary to repeat them here. The recursion relation for the edge interaction, (3.3)+(3.4)+(3.5*c*), has only one fixed point in three dimensions, governing bulk- and surface-driven edge transitions. Therefore, the various edge transitions can be uniquely labelled by the corresponding surface fixed point. For the remainder of this paper, we will adopt this convention.

In figure 3 we show the edge fixed point K_e^* as a function of the angle α at the ordinary transition. A fixed point only exists for $\alpha > \alpha^* \approx 53°$. Below α^*, K_e^* becomes imaginary, which of course does not make sense. However, the magnitude of K_e^* remains very small all the way down to $\alpha = 0°$, wherefore it is not unreasonable to

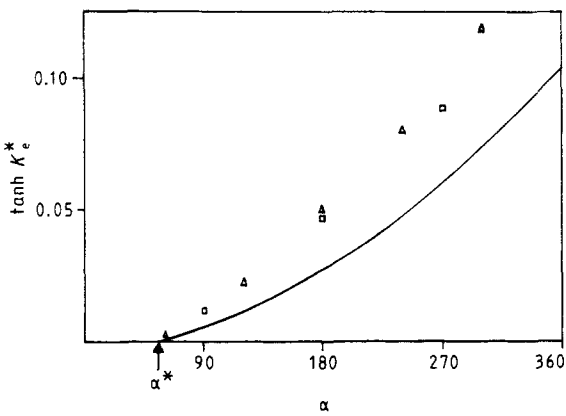

Figure 3. Edge fixed point at the ordinary transition. Solid line, MKRG; triangles, TTC RG for hexagonal lattice; squares, TTC RG for cubic lattice.

suspect that $K_e^* \equiv 0$ for all $\alpha < \alpha^*$. We should also remember that our bond-moving procedure was originally constructed only for angles greater than 60°, and perhaps it should not be trusted for smaller angles. The corresponding magnetic scaling power is plotted as the lower curve of figure 5. If we nevertheless take (3.9) seriously, it yields an increasingly irrelevant y_{he} in the region $\alpha < \alpha^*$. This means that a magnetic perturbation close to the edge does not change the critical behaviour of the edge for these values of α. Still, Cardy's scaling laws will continue to hold, and quantities like the edge magnetisation will have singularities with positive exponents. For comparison, the fixed points obtained with the TTC RG are also shown in figure 3. Apparently, this method suggests approximately the same value for α^*.

K_e^* and y_{he} at the special transition are plotted in figures 4 and 5, respectively. Here another unexpected feature emerges. For $\alpha > \alpha^{**} \approx 297°$, the edge fixed point diverges and the edge scaling power is identically equal to one for all these angles. Because y_{he} equals the dimensionality of the edge, the Nienhuis-Nauenberg (1975) criterion states that the transition should be first order. This is quite remarkable. If the special point is approached from the totally disordered phase, the edge magnetisation jumps to a non-zero value although both bulk and surface magnetisations remain zero. Also the TTC RG gave a diverging fixed point at $\alpha = 300°$.

The estimates for the fixed points are generally higher from the TTC RG than those obtained with the MKRG, in particular for the hexagonal lattice. If we extrapolate the triangle-shaped marks in figure 4 to $\tanh K_e = 1$, this appears to happen at an angle

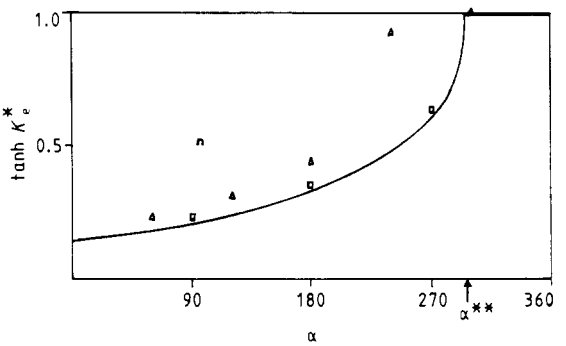

Figure 4. Edge fixed point at the special transition. Solid line, MKRG; triangles, TTC RG for hexagonal lattice; Squares, TTC RG for cubic lattice.

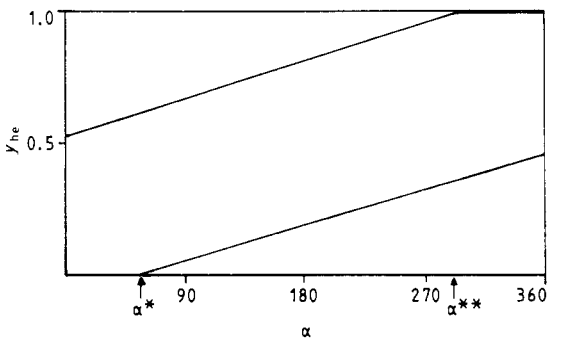

Figure 5. Edge magnetic scaling power from the MKRG at the A, special and B, ordinary transitions.

smaller than 297°. However, since Cardy showed that the magnetic scaling power is a universal function which only depends on α, α^{**} must be a universal number, too. On a cubic lattic, both methods give comparable estimates for α^{**}. This discrepancy may indicate that the identification of horizontal and vertical bonds in § 3.2 was not correct for the hexagonal lattice.

At the surface transition, where the bulk fixed point vanishes, the MKRG yields an edge fixed point which is independent of the opening angle, while the TTC RG does give a weak dependence on α. The MKRG has captured an exact property here, because the two surfaces decouple from the bulk when the bulk coupling constant vanishes. The edge is then equivalent to a special row in a two-dimensional bulk Ising model, although the lattice is bent at this row. Correspondingly, critical exponents and fixed points should take on the values of a defect row in a 2D Ising model, irrespective of α. Actually, the MKRG does reproduce the exact Onsager value $K_e^* = \frac{1}{2}\ln(1+\sqrt{2})$ on the cubic lattice. Notice also that the relations $y_{ts} = y_{te} + 1$ and $y_{hs} + 1$ are fulfilled in table 1, meaning that quantities have the same singularities both on the surface and on the edge. That this has to be the case should be obvious.

Finally, at the extraordinary transition, $K_e^* = \infty$ and $y_{he} = 1$, which is exact. The fixed points and the RG eigenvalue exponents at the different transitions are summarised in tables 1 and 2 and figures 3-5.

Table 1. Fixed points and RG eigenvalues obtained with the MKRG. Quantities denoted by — are α dependent.

	Ordinary	Special	Surface	Extraordinary
K^*	0.1398	0.1398	0	0.1398
K_s^*	0.0266	0.3394	0.4407	∞
K_e^*	—	—	0.4407	∞
y_t	0.9482	0.9482	$-\infty$	0.9482
y_{ts}	−1.6331	0.6143	0.7535	$-\infty$
y_{te}	—	—	−0.2465	$-\infty$
y_h	2.5591	2.5591	2	2.5591
y_{hs}	1.1930	1.8185	1.8814	2
y_{he}	—	—	0.8814	1

Table 2. Comparison of K_e^* obtained for different angles α from MKRG and TTC RG.

	Ordinary		Special		Surface	
α	MKRG	TTC RG	MKRG	TTC RG	MKRG	TTC RG
60°	0.42×10^{-3}	1.83×10^{-3}	0.178	0.227	0.441	0.453
120°	11.1	22.5	0.243	0.332	0.441	0.650
180° (hex)	26.6	49.4	0.339	0.476	0.441	0.650
240°	46.6	81.0	0.512	1.66	0.441	0.650
300°	71.7	119	∞	∞	0.441	0.453
90°	5.02	11.7	0.208	0.230	0.441	0.343
180° (cub)	26.6	46.5	0.339	0.350	0.441	0.441
270°	58.4	88.2	0.696	0.647	0.441	0.461

5. Conclusions

We have calculated fixed points and scaling powers within the Migdal-Kadanoff and two-terminal cluster approximations. For bulk and surface quantities, our results agree with those of Lipowsky and Wagner (1981) and Lam and Zhang (1983). Results for the edge quantities are new. The values of K_e^* and y_{he} depend on the opening angle α when the surface is at its ordinary or special fixed points, but not at the surface and extraordinary points. Two peculiarities arose: the edge fixed point disappears for $\alpha < \alpha^*$ at the ordinary transition, which we interpreted as the disappearance of independent edge behaviour, and K_e^* diverges when $\alpha > \alpha^{**}$ at the special transition. For these values of α, the edge magnetic scaling power is identically equal to one, and the otherwise irrelevant thermal exponent y_{te} becomes marginal. According to the Nienhuis-Nauenberg criterion, this means that the edge magnetisation undergoes a first-order transition, although bulk and surface magnetisations change continuously. The mechanism for this is unclear to us. One possibility is of course that the whole phenomenon is merely an artefact of our approximations. However, the fact that both the MKRG and the TTC RG yield consistent estimates for the critical angle α^{**} seems to disfavour this explanation. Furthermore, we have also performed the MKRG calculation with a finite rescaling factor for the q-state Potts model, resulting in essentially the same picture. On the other hand, the MKRG and TTC RG are similar in spirit; both methods are essentially decimations, so they might cause the same errors.

To investigate these interesting but minute effects experimentally appears difficult at present, but maybe the first-order transition at an opening angle of 300° is accessible to a Monte Carlo treatment. Since we have found that α^{**} grows with q for the q-state Potts model, the first-order transition, if it exists, will probably be easiest to find for $q = 1$ (percolation). Whether a magnetic perturbation is relevant below α^* at the ordinary transition is probably harder to test. There is no simple lattice with $\alpha < 60°$ on which to perform an experiment. Perhaps an Ising model with next-nearest-neighbour interactions could be used to investigate the case of $\alpha = 45°$.

Acknowledgment

I am grateful to John Cardy who introduced me to this subject and to the field of critical phenomena in general.

Appendix. Recursion relations for the edge coupling constant from the TTC RG

We limit ourselves to the hexagonal lattice. The recursion relation can be expressed in terms of the function $G(t_1, t_2, t_3, t_4, t_5, t_6)$, defined below. t_p, p = b, s, e, is an abbreviation for $\tanh(K_p)$.

$\alpha = 60°$:

$$t_e' = [G(t_s, t_e, t_s, t_s, t, t_s) + 2G(t_s, t_e, 0, t_s, 0, 0) + 3G(0, t_e, 0, 0, 0, 0]/6$$

$\alpha = 120° - 240°$:

$$t_e' = [2G(t, t_e, t_s, t, t, t_s) + pG(t, t_e, t, t, t, t) + 2G(t_s, t_e, 0, t_s, 0, 0) + (2-p)G(0, t_e, 0, 0, 0, 0)]/6$$

where $p = 0$ if $\alpha = 120°$, $p = 1$ if $\alpha = 180°$ and $p = 2$ if $\alpha = 240°$.

$\alpha = 300°$:

$$t'_e = [G(t_s, t_e, t_s, t_s, 0, t_s) + 2G(t, t_e, t_s, t, t, t_s) + 3G(t, t_e, t, t, t, t)]/6.$$

The function G is computed using the break-collapse method

$$G(t_1, t_2, t_3, t_4, t_5, t_6) = N/D$$

where N and D are defined by the following set of equations:

$$N_A(x, y, z) = x^2 + y^2 + z^2 + x^2y^2z^2$$

$$D_A(x, y, z) = 1 + x^2y^2 + y^2z^2 + z^2x^2$$

$$N_B(x, y, z) = x^2(1 + yz)^2 + (y + z)^2$$

$$D_B(x, y, z) = (1 + yz)^2 + x^2(y + z)^2$$

$$N_c(x, y, z) = (x + y + z + xyz)^2$$

$$D_c(x, y, z) = (1 + xy + yz + zx)^2$$

$$N_1 = (1 - t_4)N_A(t_1, t_2, t_3) + t_4N_B(t_2, t_1, t_3)$$

$$N_2 = (1 - t_4)N_B(t_1, t_2, t_3) + t_4N_c(t_1, t_2, t_3)$$

$$N_3 = (1 - t_4)N_B(t_3, t_1, t_2) + t_4N_C(t_1, t_2, t_3)$$

$$N_4 = N_C(t_1, t_2, t_3)$$

$$N_5 = (1 - t_5)N_1 + t_5N_2$$

$$N_6 = (1 - t_5)N_3 + t_5N_4$$

$$N = (1 - t_6)N_5 + t_6N_6.$$

D_i, $i = 1$-6, and D are defined in the same way as the corresponding N_i, with all N replaced by D.

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